ADVANCES IN MATHEMATICS 39, 1–30 (1981)

Resolutions of Determinantal Ideals: The Submaximal Minors

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1. INTRODUCTION

For many years there has been considerable interest in finding a resolution of the ideal generated by the minors of order p of a generic $m \times n$ matrix. To put the problem more precisely, suppose R_0 is a commutative ring and X_{ij} are variables with $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$. If we let $R = R_0[X_{ij}]$ be the polynomial ring over R_0, then we have the "generic" matrix (X_{ij}) and we may form the ideal I_p in R generated by the $p \times p$ minors of this matrix. The problem, then, is to find an explicit free resolution of the ideal I_p over the ring R. It was proved by Eagon and Hochster [10] that R/I_p has a resolution of length $(m-p+1)(n-p+1)$, but their proof consisted in showing that the ideal I_p is perfect; it did not provide a construction of the resolution. In fact, it is not known whether the Betti numbers of the ideal I_p depend on the characteristic of the ground ring R_0.

In [12], Lascoux succeeded in giving an explicit resolution provided that the ground ring R_0 contained the field, Q, of rational numbers. His construction rests heavily on the theory of Schur functors and the fact that in characteristic zero the Schur functors are the irreducible representations of the general linear group. Over the integers, however, the construction breaks down despite the fact that one can define the Schur functors over an arbitrary commutative ring (see [1, 2, 14, 15] for various constructions of Schur functors).

In analyzing the work of Lascoux and its subsequent reworking by Nielsen [13], some basic facts seemed to clamor for attention. One was that within a resolution of R/I_p, there appeared to be two types of boundary maps: one of degree 1 and one of degree p. The maps of degree 1 were maps between sums of Schur functors of fixed Durfee square k (see Section 2 for definitions), while the maps of degree p were from sums of Schur functors of

* The authors were partially supported by NSF Grant MCS 77-04951.

0001-8708/81/010001-30$05.00/0

Durfee square k to others of Durfee square $k-1$. It therefore seems necessary to define complexes

$$\mathbb{X}(k): \cdots \to X_l(k) \xrightarrow{d_l(k)} X_{l-1}(k) \to \cdots,$$

where k denotes the size of the Durfee square and $d_l(k)$ is a map of degree 1, and to define maps of complexes

$$\phi(k): \mathbb{X}(k) \to \mathbb{X}(k-1),$$

where $\phi(k)$ has degree p. The resulting double complex should then provide the desired resolution. (Recall that we are talking about resolutions over the graded polynomial ring $R_0[X_{ij}]$, so that the degree of a map makes sense.)

Using the Lascoux and Nielsen results in characteristic zero as a guide, it also seems that the modules $X_l(k)$ defined, say, over Z (the integers) should be Z-forms of the corresponding terms in the Lascoux resolution (see Section 2 for a definition of Z-forms). Using explicit calculations as a guide, it appears that the Z-forms cannot be a straightforward parroting of the construction in characteristic zero. In fact, it is easy to see that $X_2(1)$ cannot be the sum of Schur functors of the Lascoux resolution. Moreover, the maps $\phi(k)$ of degree p in characteristic zero always have determinantal coefficients while over the integers such a restriction on the map $\phi(2)$ would probably make it impossible to construct a resolution.

In [4] we started the program of applying the foregoing observations to the case when $p = n \leqslant m$, i.e., the maximal order minors. For that case, the only Durfee squares are of size 1 so that we needed to construct only the complex $(\mathbb{X}(1), d(1))$. This case being successfully concluded, we next tackled the case: $m \geqslant n = p+1$. What we do in this paper is prove that the program outlined above works in this submaximal case. As a corollary, we see that the Betti numbers of I_p for a generic $mx(p+1)$ matrix are independent of the characteristic.

Before outlining the rest of this paper, it may be of interest to note that in 1890, Hilbert's paper [11], in which he proved, among other things, the syzygy theorem, was thought to have killed off invariant theory. However, the attempt to find the syzygies of the ideal I_p has been one of the factors to revive interest in and lead to generalizations of classical invariant theory.

The construction of our resolution proceeds as follows: We first construct a complex

$$\mathbb{X}^p(1): \cdots \to X_4^p(1) \to X_3^p(1) \to X_2^p(1) \to X_1^p(1) \to R$$

consisting of universally free modules, and with boundary maps of degree 1. This is done in Section 3, using various auxiliary constructions.

The next step is to introduce Schur complexes (defined in Section 4) attached to maps of free modules, a characteristic-free construction whose characteristic zero counterpart is given by Nielsen in [13]. In Section 5 we show that the Schur complex corresponding to the partition

$$\underbrace{(p+1,\ldots,p+1)}_{r}$$

yields a resolution of I_{p+1}^r when $p+1$ is the size of the maximal order minors, and in particular yields a resolution

$$\mathbb{X}^p(2)\colon \cdots > X_5^p(2) \to X_4^p(2) \to I_{p+1}^2 \to 0$$

of I_{p+1}^2 when $r=2$. We also show that $I_{p+1}^2 \subset H_3(\mathbb{X}(1))$ and that $H_i(\mathbb{X}(1))=0$ for $i>3$ and $i=1,2$. These facts imply the existence of a map

$$\phi(2)\colon \mathbb{X}^p(2) \to \mathbb{X}^p(1)$$

of degree p, and the mapping cone of this map of complexes provides a free complex over R/I_p. The final step in showing that this mapping cone is a resolution of R/I_p consists in applying the acyclicity lemma [5]. After inverting a $p\times p$ minor it is easy to see that $H_3(\mathbb{X}^p(1))=I_{p+1}^2$ and this is all that is needed to complete the proof.

2. Preliminaries

If R is a commutative ring, and F is a free R-module, we may consider the morphism $\alpha\colon F\otimes \Lambda^k F \to \Lambda^2 F\otimes \Lambda^{k-1}$ obtained by composing the maps

$$F\otimes \Lambda^k F \xrightarrow{1\otimes\Delta} F\otimes F\otimes \Lambda^{k-1}F \xrightarrow{m\otimes 1} \Lambda^2 F\otimes \Lambda^{k-1}F,$$

where $\Delta\colon \Lambda^k F\to F\otimes\Lambda^{k-1}F$ is the appropriate component of the diagonal map of $\Lambda F\to \Lambda F\otimes \Lambda F$, and $m\colon \Lambda F\otimes \Lambda F\to \Lambda F$ is the multiplication map in the exterior algebra, ΛF. If we denote the kernel of α by $T_R(F)$, we see that $T_R(F)$ is a functor on the category of free R-modules. If $\phi\colon R\to S$ is a morphism of rings, we have the functor

$$E_\phi\colon F\to S\otimes_R F$$

from the category of free R-modules to the category of free S-modules and a natural transformation of functors

$$T_R \to T_s E_\phi.$$

If $R = Z$, the ring of integers, it is easily seen that $T_Z(F) = 0$ for all F, but if $R = Z/(2)$, then $T_{Z/2}(F) \neq 0$ for all free $Z/(2)$-modules F. Thus it is clear that $T_{Z/(2)} E_\phi \neq Z/(2) \otimes T_Z$ where $\phi: Z \to Z/(2)$ is the obvious ring morphism.

On the other hand, if we define $T_R(F_1, F_2)$ to be the functor (of two free variables) $F_1 \otimes_R F_2$, and if we define $E_\phi(F_1, F_2)$ to be $(S \otimes_R F_1, S \otimes_R F_2)$ for a ring morphism $\phi: R \to S$, then we again have the natural transformation

$$T_R \to T_s E_\phi$$

and in this case, $T_s E_\phi = S \otimes_R T_R$.

These examples lead us to make the following definition:

DEFINITION 2.1. Let T_R $(F_1, ..., F_n)$ be a functor defined for all commutative rings R and all n-tuples of free R-modules $F_1, ..., F_n$. $T_R(F_1, ..., F_n)$ is called a *universally free functor* if

(a) $T_R(F_1, ..., F_n)$ is a free R-module;

(b) if $\phi: R \to S$ is a ring morphism, then $S \otimes_R T_R$ is naturally equivalent to the functor $T_s E_\phi$ (where now E_ϕ is the functor sending $(F_1, ..., F_n) \to (S \otimes_R F_1, ..., S \otimes_R F_n)$).

Consider the functors $T_R(F)$ and $T'_R(F)$ where $T_R(F) = S_2(F)$ and $T'_R(F) = D_2(F)$, $S_2(F)$ denoting the second symmetric power and $D_2(F)$ the second divided power of F. Both of these functors are universally free and are naturally equivalent when R contains a field of characteristic zero.

DEFINITION 2.2. Let T'_R, T_R and T''_R be universally free functors. We say that T_R and T''_R are *Z-forms of* T_R if T'_Q and T''_Q are naturally equivalent to T_Q, where Q is the field of rational numbers.

We will now review the basic definitions of an important class of universally free functors, namely the skew Schur and co-Schur functors. A detailed exposition of these constructions will be found in [1, 2, 15], so we will omit the proofs of universal freeness here.

DEFINITION 2.3. Let n be a positive integer. *A partition of weight n* is a decreasing sequence of positive integer $\lambda = (\lambda_1, ..., \lambda_q)$ with $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_q \geqslant 1$ such that $\lambda_1 + \cdots + \lambda_q = n$. We denote the *weight*, n, of λ by $|\lambda|$, and call q the height of λ. To each partition, λ, of weight n, we associate its *transpose* $\tilde{\lambda} = (\tilde{\lambda}_1 \cdots \tilde{\lambda}_t)$, where $\tilde{\lambda}_k$ is the number of integers λ_l such that $\lambda_l \geqslant k$. If $\mu = (\mu_1, ..., \mu_p)$ is also a partition, we will say that μ is a *sub-partition* of λ, or that $\mu \subset \lambda$, if $p \leqslant q$ and $\mu_i \leqslant \lambda_i$ for $i = 1, ..., p$.

To the partition λ we may associate the set $\mathscr{S}_\lambda$ of integral points in the plane: $\mathscr{S}_\lambda = \{(x, u)/0 \leqslant u \leqslant q-1,\ 0 \leqslant x \leqslant \lambda_{q-u}-1$, and x and u are integers$\}$. The top row of this set has λ_1 elements, the second row from the top has λ_2 elements, etc. In the transpose $\tilde{\lambda}$, the integer $\tilde{\lambda}_1$ is the number of points in the left-hand column of the set $\mathscr{S}_\lambda$, the integer $\tilde{\lambda}_2$ is the number of points in the next column over, etc. Thus we have

$$\tilde{\lambda}_1 = q \geqslant \tilde{\lambda}_2 \cdots \geqslant \tilde{\lambda}_t,\ t = \lambda_1, \qquad \text{and} \qquad \tilde{\lambda}_1 + \cdots \tilde{\lambda}_t = n.$$

Therefore $\tilde{\lambda}$ is also a partition of weight n and $\tilde{\tilde{\lambda}} = \lambda$.

Let F be a free R-module and let $\alpha = (\alpha_1 \ldots, \alpha_q)$ be a sequence of non-negative integers. We define $\Lambda_\alpha(F)$ to be $\Lambda^{\alpha_1}F \otimes_R \cdots \otimes_R \Lambda^{\alpha_q}F$; $S_\alpha(F) = S_{\alpha_1}F \otimes_R \cdots \otimes_R S_{\alpha_q}F$ and $D_\alpha(F) = D_{\alpha_1}F \otimes_R \cdots \otimes_R D_{\alpha_q}F$, where ΛF, SF and DF are the kth exterior, symmetric and divided power modules of F. Given the module F and two partitions

$$\lambda = (\lambda_1, \ldots, \lambda_q), \qquad \mu = (\mu_1, \ldots, \mu_{q'})$$

with $\mu \subset \lambda$, we have the sequences $\lambda/\mu = (\lambda_1 - \mu_1, \ldots, \lambda_{q'} - \mu_{q'}, \lambda_{q'+1}, \ldots, \lambda_q)$ and $\tilde{\lambda}/\tilde{\mu} = (\tilde{\lambda}_1 - \tilde{\mu}_1, \ldots, \tilde{\lambda}_{t'} - \tilde{\mu}_{t'},\ \tilde{\lambda}_{t'+1}, \ldots, \tilde{\lambda}_t)$ where $\tilde{\lambda} = (\tilde{\lambda}_1, \ldots, \tilde{\lambda}_t)$ and $\tilde{\mu}_1, \ldots, \tilde{\mu}_{t'}$). We want to define a map

$$\sigma_{\lambda/\mu}(F) \colon \Lambda_{\lambda/\mu}(F) \to S_{\tilde{\lambda}/\tilde{\mu}}(F).$$

In order to define this map recursively, we introduce some auxiliary notation. First, we will arbitrarily call the sequence (0) a partition contained in every partition λ. We assign to it weight 0 and height 0. In this way, $\lambda/(0) = \lambda$ and $\tilde{\lambda}/(\tilde{0}) = \tilde{\lambda}$.

Next if λ is any partition other than (0), and $\lambda = (\lambda_1, \ldots, \lambda_q)$, $\tilde{\lambda} = (\tilde{\lambda}_1, \ldots, \tilde{\lambda}_t)$, we denote by $\lambda - 1$ the partition $(\lambda'_1, \ldots, \lambda'_p)$ where $p = \tilde{\lambda}_2$ and $\lambda'_k = \lambda_k - 1$ for $k = 1, \ldots; p$. $\lambda - 1$ is clearly a partition whose weight is $|\lambda| - q$ and whose transpose, $\widetilde{\lambda - 1}$, is $(\tilde{\lambda}_2, \ldots, \tilde{\lambda}_t)$. If $\mu \subset \lambda$, then $\mu - 1 \subset \lambda - 1$, and $\widetilde{\lambda - 1}/\widetilde{\mu - 1} = (\tilde{\lambda}_2 - \tilde{\mu}_2, \ldots, \tilde{\lambda}_{t'} - \tilde{\mu}_{t'}, \tilde{\lambda}_{t'+1}, \ldots, \tilde{\lambda}_t)$. If $\mu = (0)$, we define $\mu - 1$ to be μ again.

Notice that if $\mu = (\mu_1, \ldots, \mu_{q'}) \subset \lambda = (\lambda_1, \ldots, \lambda_q)$ and $q' = q$, then $\Lambda_{\lambda/\mu}(F) = \Lambda_{\lambda - 1/\mu - 1}(F)$ and $S_{\tilde{\lambda}/\tilde{\mu}}(F) = S_{\widetilde{\lambda-1}/\widetilde{\mu-1}}(F)$. We may therefore proceed to peel off columns from the partitions λ and μ until we arrive at the situation where λ has more rows than μ has, i.e., $q' < q$. In this case we want to define

$$\begin{aligned}\sigma_{\lambda/\mu} \colon & \Lambda^{\lambda_1 - \mu_1}F \otimes \cdots \otimes \Lambda^{\lambda_{q'} - \mu_q}F \\ & \otimes \Lambda^{\lambda_{q'+1}}F \otimes \cdots \otimes \Lambda^{\lambda_q}F \to S_{\tilde{\lambda}_1 - \tilde{\mu}_1}F \otimes \cdots \otimes S_{\tilde{\lambda}_{t'} - \tilde{\mu}_{t'}}F \otimes \cdots \otimes S_{\tilde{\lambda}_t}F.\end{aligned}$$

We have the map

$$\eta: \Lambda^{\lambda_{q'+1}}F \otimes \cdots \otimes \Lambda^{\lambda_q}F \to F \otimes \Lambda^{\lambda_{q'+1}-1}F \otimes \cdots \otimes F \otimes \Lambda^{\lambda_q - 1}F$$
$$\to S_{q-q'}F \otimes \Lambda^{\lambda_{q'+1}-1}F \otimes \cdots \otimes \Lambda^{\lambda_q-1}F$$

obtained by diagonalizing each of the $q-q'$ factors

$$\Lambda^{\lambda_k}F \to F \otimes \Lambda^{\lambda_k - 1}F$$

and then multiplying the $q-q'$ F's to $S_{q-q'}F$. Notice that $q-q' = \tilde{\lambda}_1 - \tilde{\mu}_1$. Tensoring the map η with the identity on $\Lambda^{\lambda_1-\mu_1}F \otimes \cdots \otimes \Lambda^{\lambda_q - \mu_q}$, we get a map

$$\Lambda_{\lambda/\mu}(F) \to S_{\tilde{\lambda}_1 - \tilde{\mu}_1}F \otimes \Lambda_{\lambda-1/\mu-1}F.$$

Assuming that we already know the map $\sigma_{\lambda'/\mu'}(F)$ for all partitions λ' of weight less than $|\lambda|$, we have the map

$$\sigma_{\lambda-1/\mu-1}: \Lambda_{\lambda-1/\mu-1}(F) \to S_{\widetilde{\lambda-1}/\widetilde{\mu-1}}(F).$$

The map $\sigma_{\lambda/\mu}$ is now defined as the composition:

$$\Lambda_{\lambda/\mu}(F) \to S_{\tilde{\lambda}_1-\tilde{\mu}_1}(F) \otimes \Lambda_{\lambda-1/\mu-1}(F)$$
$$\to S_{\tilde{\lambda}_1-\tilde{\mu}_1}(F) \otimes S_{\widetilde{\lambda-1}/\widetilde{\mu-1}}(F) = S_{\tilde{\lambda}/\tilde{\mu}}(F).$$

To complete this recursive definition, it suffices to define $\sigma_{\lambda/\mu}$ when $\lambda = (0)$. In this case, $\mu = (0)$ also, $\Lambda_{\lambda/\mu}(F) = R$, $S_{\lambda/\mu}(F) = R$, and we define $\sigma_{\lambda/\mu}(F)$ to be the identity.

In a similar way, we may define a map

$$\tau_{\lambda/\mu}(F): D_{\lambda/\mu}(F) \to \Lambda_{\tilde{\lambda}/\tilde{\mu}}(F).$$

For convenience, we may think of a partition as a non-increasing finite sequence of non-negative integers. With this convention, we identify a partition $(\lambda_1,\ldots,\lambda_s)$ with the partition $(\lambda_1,\ldots,\lambda_s,0)$.

DEFINITIONS 2.4. If F is a free R-module and λ, μ are partitions with $\mu \subset \lambda$, we define

$$L_{\lambda/\mu}(F) = \text{Image } \sigma_{\lambda/\mu}(F),$$
$$K_{\lambda/\mu}(F) = \text{Image } \tau_{\lambda/\mu}(F).$$

$L_{\sigma/\mu}(F)$ is called the *skew Schur functor* corresponding to the pair of partitions $\mu \subset \lambda$, and $K_{\sigma/\mu}(F)$ is called the *skew co-Schur functor* corresponding to the pair of partitions $\mu \subset \lambda$. If $\mu = (0)$, we denote $L_{\lambda/\mu}(F)$ by $L_\lambda(F)$, and $K_{\lambda/\mu}(F)$ by $K_\lambda(F)$.

It is proved in [2] that the skew Schur and co-Schur functors are universally free.

DEFINITION 2.5. Let $\lambda = (\lambda_1, ..., \lambda_q)$ be a partition other than (0). We say that λ has *Durfee square* k if $\lambda_k \geqslant k$ but $\lambda_{k+1} \leqslant k$. We arbitrarily assign (0) Durfee square 0.

From this definition, we see that $(\lambda_1, ..., \lambda_q)$ has Durfee square 1 if and only if $\lambda_2 = \cdots = \lambda_q = 1$. Such a partition is called a *hook*. It is clear that for any partition λ, λ and $\tilde{\lambda}$ have the same Durfee square.

Two other formal ideas will arise in the sequel: linear complexes and linearly exact complexes. Suppose $R = R_0 \oplus R_1 \oplus \cdots$ is a graded (commutative) ring. We will say that an R-module C is *induced* if $C = R \otimes_{R_0} M$ where M is an R_0-module. If M_1 and M_2 are R_0-modules, and if $\phi_0 : M_1 \to R_1 \otimes_{R_0} M_2$ is an R-morphism, then ϕ_0 induces an R-morphism $\phi : C_1 \to C_2$ where C_i are the induced R-modules $R \otimes_{R_0} M_i$. The morphism ϕ on $R_k \otimes_{R_0} M_1$ is simply

$$R_k \otimes M_1 \to R_k \otimes R_1 \otimes M_2 \to R_{k+1} \otimes M_2,$$

where $R_k \otimes R_1 \to R_{k+1}$ is multiplication in R. The map ϕ is said to be *linear* or of *degree one*.

DEFINITION 2.6. Let R be a graded ring as above, and let $\mathbb{C}: \cdots \to C_k \to^{d_k} C_{k-1} \to \cdots \to C_1 \to^{d_1} C_0$ be a complex of R-modules. We say that $\mathbb{C}$ is a *linear* or *degree one complex* if each C_k is an induced module, i.e., $C_k = R \otimes_{R_0} M_k$ and $d_k : C_k \to C_{k-1}$ is of degree one. We say that $\mathbb{C}$ is *linearly exact* if for all $k \geqslant 2$, the sequence

$$0 \to M_k \xrightarrow{(d_k)_0} R_1 \otimes M_{k-1} \xrightarrow{(d_{k-1})_1} R_2 \otimes M_{k-2}$$

is exact.

For the sake of convenience, we will make one further convention. If $\mathbb{C}$ and $\mathbb{C}'$ are complexes and $\phi = \{\phi_k : C_k \to C'_k\}$ is a family of maps from the chains of $\mathbb{C}$ to those of $\mathbb{C}'$, we will call ϕ a *map of complexes* if each square

$$\begin{array}{ccc} C_k & \longrightarrow & C_{k-1} \\ {\scriptstyle \phi_k}\downarrow & & \downarrow{\scriptstyle \phi_{k-1}} \\ C'_k & \longrightarrow & C'_{k-1} \end{array}$$

either commutes or anticommutes.

3. The Fundamental Durfee Square One Complexes

In this section, R is a commutative ring, F and G are free R-modules of ranks m and n, respectively, and $\phi: F \otimes G \to R$ is an R-map. We will identify $\mathrm{Hom}_R(F \otimes G, R)$ with $\mathrm{Hom}_R(G, F^*)$ via the canonical isomorphism. With this identification, we will use the same symbol ϕ to denote the corresponding map $\phi: G \to F^*$, and write $\phi^*: F \to G^*$ for the map dual to ϕ. We will also denote by c_ϕ the element of $F^* \otimes G^*$ corresponding to ϕ under the canonical isomorphism:

$$\mathrm{Hom}_R(F \otimes G, R) \approx F^* \otimes G^*.$$

Definition 3.1. $\mathbb{A}(\phi)$ is the (doubly graded) complex $(\Lambda F^* \otimes DG, \partial_\phi)$ where the differential ∂_ϕ is given by the action of $c_\phi \in \Lambda F^* \otimes SG^*$ on $\Lambda F^* \otimes DG$. The subcomplex

$$0 \to D_l G \to F^* \otimes D_{l-1} G \to \cdots \to \Lambda^{l-i} F^* \otimes D_i G \to \cdots \to \Lambda^l F^* \to 0$$

of $\mathbb{A}(\phi)$ will be denoted by $\mathbb{A}^l(\phi)$. The component of $\mathbb{A}^l(\phi)$ of degree i, denoted by $A^l_i(\phi)$, is the term $\Lambda^{l-i} F^* \otimes D_i G$.

We note that $\mathbb{A}^l(\phi)$ is isomorphic to the complex

$$0 \to \Lambda^m F \otimes D_l G \to \Lambda^{m-1} F \otimes D_{l-1} G \to \cdots \to \Lambda^{m-l+i} F \otimes D_i G \\ \to \cdots \to \Lambda^{m-l} F \to 0,$$

the isomorphism being induced by $\Lambda^m F \otimes \Lambda^{l-i} F^* \approx \Lambda^{m-l+i} F$. We will use the same notation, $\mathbb{A}^l(\phi)$, to denote either one of these two isomorphic complexes.

Definition 3.2. $\mathbb{B}(\phi)$ is the (doubly graded) complex $(DF^* \otimes \Lambda G, \delta_\phi)$, where the differential δ_ϕ is given by the action of $c_\phi \in DF^* \otimes \Lambda G^*$ on $DF^* \otimes \Lambda G$. The subcomplex

$$0 \to \Lambda^l G \to F^* \otimes \Lambda^{l-1} G \to \cdots \to D_i F^* \otimes \Lambda^{l-i} G \to \cdots \to D_l F^* \to 0$$

of $\mathbb{B}(\phi)$ will be denoted by $\mathbb{B}_l(\phi)$. The component of degree i of $\mathbb{B}_l(\phi)$, denoted by $B^i_l(\phi)$, is the term $D_i F^* \otimes \Lambda^{l-i} G$.

Definition 3.3. The tensor product of the complexes $\mathbb{A}^{m-p}(\phi)$ and $\mathbb{A}^{n-p}(\phi^*)$ is again a complex, $\mathbb{A}^{m-p}(\phi) \otimes \mathbb{A}^{n-p}(\phi^*)$, with its customary boundary map. We denote by $\mathbb{U}^p(\phi)$ this complex with its degree shifted by one, i.e., $\mathbb{U}^p(\phi) = \{U^p_{k+1}(\phi)\}$ with

$$U^p_{k+1}(\phi) = (\mathbb{A}^{m-p}(\phi) \otimes \mathbb{A}^{n-p}(\phi^*))_k \qquad \text{for} \quad k \geqslant 0.$$

We will denote the boundary map of $\mathbb{U}^p(\phi)$ by ∂^u_ϕ.

Explicitly, we have

$$U^p_{k+1}(\phi) = \sum_{a+b=k} \Lambda^{p+a}F \otimes D_aG \otimes \Lambda^{p+b}G \otimes D_bF.$$

To describe the boundary map explicitly, we let $\{f_i\}$, $i = 1,..., m$, and $\{g_j\}$, $j = 1,..., n$, be bases of F and G, respectively, and let $\{\rho_i\}$, $\{\gamma_j\}$ be their respective dual bases. Then for $x \otimes y \otimes u \otimes v \in \Lambda^{p+a}F \otimes D_aG \otimes \Lambda^{p+b}G \otimes D_bF$ we have:

$$\partial^u_\phi(x \otimes y \otimes u \otimes v) = \sum \phi(f_i \otimes g_j)\{\rho_i(x) \otimes \gamma_j(y) \otimes u \otimes v + (-1)^a x \otimes y \otimes \gamma_j(u) \otimes \rho_i(v)\}.$$

DEFINITION 3.4. Let $1_F: F \to F$ be the identity map. Instead of writing $\mathbb{A}^l(1_F)$ or $\mathbb{B}_l(1_F)$ we shall denote these complexes by $\mathbb{A}^l(F)$ and $\mathbb{B}_l(F)$, and write ∂_F, δ_F for their boundary maps. Notice that $\mathbb{A}^l_i(F) = \mathbb{B}^i_l(F) = \Lambda^{l-i}F \otimes D_iF$.

PROPOSITION 3.5. *For any free R-module F we have:*

(1) *$\mathbb{A}^l(F)$ is exact for $l > 0$, i.e., $H_i(\mathbb{A}^l(F)) = 0$ for $i \geqslant 0$.*

(2) *$H^i(\mathbb{B}_l(F)) = 0$ for $i \leqslant [(l-1)/2]$.*

(3) *$\operatorname{Coker}(B^i_l(F) \to B^{i+1}_l(F))$ is free for $i \leqslant [(l-1)/2]$.*

Proof. The proof of (1) is well known (see [6]). To prove (2), we choose a basis $\{f_1,..., f_m\}$ for F, and let $\{\rho_1,..., \rho_m\}$ be the dual basis for F^*. Then the set $\{f_1^{\alpha_1} \wedge \cdots \wedge f_m^{\alpha_m} \otimes f_1^{(\beta_1)} \cdots f_m^{(\beta_m)} / \alpha_j = 0 \text{ or } 1,\ \beta_i \geqslant 0,\ \sum_{j=1}^m \alpha_j + \beta_j = l\}$ is a basis for $\mathbb{B}_l(F)$. (By $f_i^{(\beta_i)}$ we mean the β_ith divided power of f_i in $D(F)$). The *content* of the basis element $f_1^{\alpha_1} \wedge \cdots \wedge f_m^{\alpha_m} \otimes f_1^{(\beta_1)} \cdots f_m^{(\beta_m)}$ is the m-tuple $(\alpha_1 + \beta_1,..., \alpha_m + \beta_m)$.

Given an m-tuple $\gamma = (\gamma_1,..., \gamma_m)$ of non-negative integers such that $\gamma_1 + \cdots + \gamma_m = l$, we can consider the submodule of $\mathbb{B}_l(F)$ generated by all basis elements with content γ. It is easy to see that this submodule forms a subcomplex of $\mathbb{B}_l(F)$ which we call the *strand of content* γ and will denote by $(\mathbb{B}_l(F))_\gamma$. To complete the proof of (2), we need the following lemma:

LEMMA 3.6. *Let $\gamma = (\gamma_1,..., \gamma_m)$ be such that $\gamma_1 + \cdots + \gamma_m = l$, and $\gamma_k = 1$ for some k. Then the complex $(\mathbb{B}_l(F))_\gamma$ is split exact, i.e.,*

$$0 \to B^0_l(F)_\gamma \to B^1_l(F)_\gamma \to \cdots \to B^l_l(F)_\gamma \to 0$$

is a split exact sequence.

Proof. We will show that the identity map of $(\mathbb{B}_l(F))_\gamma$ is chain homotopic to zero. That is, we will construct R-maps $S_i\colon (B_l^i(F))_\gamma \to (B_l^{i-1}(F)_\gamma$ such that $\delta_F S_{i-1} + S_i \delta_F = 1$ on $(\mathbb{B}_l(F))_\gamma$. For $x \otimes y \in (B_l^i(F))_\gamma \subset D_i F \otimes \Lambda^{l-i} F$, define $S_i(x \otimes y) = \rho_k(x) \otimes f_k \wedge y$. By choosing $x \otimes y$ to be a basis element of $(\mathbb{B}_l(F))_\gamma$, it is easy to see that $\{S_i\}$ is indeed a contracting homotopy (keeping in mind the fact that $\gamma_k = 1$); in fact $\{S_i\}$ is a splitting homotopy.

We now return to the proof of Proposition 3.5. Clearly, $\mathbb{B}_l(F)$ is the direct sum, $\sum_\gamma (B_l(F))\gamma$ of the subcomplexes $\mathbb{B}_l(F)_\gamma$. Thus $H^i(\mathbb{B}_l(F)) = \sum_\gamma H^i((\mathbb{B}_l(F))\gamma)$. Observe that if $\gamma = (\gamma_1,..., \gamma_m)$ is such that $\gamma_k \neq 1$ for all $k = 1,..., m$, then $(B_l^i(F))_\gamma = 0$ for $i \leqslant [(l-1)/2]$. For if $f_1^{\alpha_1} \wedge \cdots \wedge f_m^{\alpha_m} \otimes f_1^{(\beta_1)} \cdots f_m^{(\beta_m)}$ has content γ, and $\gamma_k \neq 1$ for all k, then $\beta_k = 0$ implies $\alpha_k = 0$. Thus $\sum \alpha_k \leqslant \sum \beta_k$ since $\alpha_k = 0$ or 1. However, if $i \leqslant [(l-1)/2]$, we have $\sum \beta_k \leqslant [(l-1)/2]$ and $\sum \alpha_k \geqslant [(l+1)/2]$ which is impossible. We see, therefore, that for $i \leqslant [(l-1)/2]$

$$H^i(\mathbb{B}_l(F)) = \sum_{\gamma'} H^i(\mathbb{B}_l(F))_{\gamma'},$$

where γ' runs over those contents with $\gamma'_k = 1$ for at least one k, and this last sum is 0 by Lemma 3.6.

To prove (3), consider the sequence

$$0 \to B_l^0(F) \to B_l^1(F) \to \cdots \to B_l^t(F) \to B_l^{t+1}(F) \to L \to 0, \tag{S}$$

where $t = [(l-1)/2]$ and L is the cokernel of $B_l^t(F) \to B_l^{t+1}(F)$. This sequence is exact (by (2)) and each of the modules $B_l^i(F)$ is universally free for $i = 0,..., t+1$. Therefore, for $i < [(l-1)/2]$ it is clear that $\operatorname{Coker}(B_l^i(F) \to B_l^{i+1}(F))$ is universally free. It remains only to show that L is free. To do this it suffices to prove it when R is the ring of integers. But the split exactness of the sequence of $B_l^i(F)$ guarantees the exactness of (S) when localized at Q and also when reduced modulo p for every rational prime p. Thus the rank of L is constant and L is therefore (universally) free. This completes the proof of Proposition 3.5.

$U_k^p(\phi)$ was defined to be the sum

$$\sum_{a+b=k} \Lambda^{p+a} F \otimes D_a G \otimes \Lambda^{p+b} G \otimes D_b F$$

If $x \otimes y \otimes u \otimes v \in \Lambda^{p+a} F \otimes D_a G \otimes \Lambda^{p+b} G \otimes D_b F$ with $a + b = k$, then $x \otimes v \in \mathbb{A}_b^{p+k}(F)$ and $u \otimes y \mathbb{A}^{p+k}(G)$. Thus we may apply the map ∂_F to $(x \otimes v)$ and we have $\partial_F(x \otimes v) \otimes y \otimes u \in \Lambda^{p+a+1} F \otimes D_{b-1} F \otimes D_a G \otimes \Lambda^{p+b} G \approx \Lambda^{p+a+1} F \otimes D_a G \otimes \Lambda^{p+b} G \otimes D_{b-1} F$. But $\Lambda^{p+a+1} F \otimes D_a G \otimes \Lambda^{p+b} G \otimes D_{b-1} F$ is in $U_k^{p+1}(\phi)$ so, with this identification understood, we may consider $\partial_F(x \otimes v) \otimes (y \otimes u)$ to be an element of $U_k^{p+1}(\phi)$. Similarly, $u \otimes y \in \mathbb{A}_a^{p+k}(G)$, we can apply ∂_G to $u \otimes y$, and after rearrangement of terms we may consider $(x \otimes v) \otimes \partial_G(u \otimes y)$ to be an element of $U_k^{p+1}(\phi)$. In exactly the same way, noting that $x \otimes v \in \mathbb{B}_{p+k}^b(F)$ and $u \otimes y \in \mathbb{B}_{p+k}^a(G)$, we may consider the elements $\delta_F(x \otimes v) \otimes y \otimes u$ and $(x \otimes v) \otimes \delta_G(u \otimes y)$ as elements of $U_{k+2}^{p-1}(\phi)$. With these conventions in mind, we make the following definitions:

Definitions 3.7. We define the maps

$$\partial_{k+1}^p(F, G)\colon U_{k+1}^p(\phi) \to U_k^{p+1}(\phi)$$

and

$$\delta_{k+1}^p(F, G)\colon U_{k+1}^p(\phi) \to U_{k+2}^{p-1}(\phi)$$

as follows. If $x \otimes y \otimes u \otimes v \in \Lambda^{p+a}F \otimes D_aG \otimes \Lambda^{p+b}G \otimes D_bF$,

$$\begin{aligned}\partial_{k+1}^p(F, G)(x \otimes y \otimes u \otimes v) &= (x \otimes v) \otimes \partial_G(u \otimes y)\\ &\quad + (-1)^a\, \partial_F(x \otimes v) \otimes (y \otimes u)\\ \delta_{k+1}^p(F, G)(x \otimes y \otimes u \otimes v) &= (x \otimes v) \otimes \delta_G(u \otimes y)\\ &\quad + (-1)^{a+1}\, \delta_F(x \otimes v) \otimes (y \otimes u).\end{aligned}$$

The notation $\partial(F, G)$, $\delta(F, G)$ underscores the fact that these maps depend only on the modules F and G; they are completely independent of the map ϕ.

Proposition 3.8.

(1) $U_{k+2}^{p-1}(\phi) \xrightarrow{\partial_{k+2}^{p-1}(F,G)} U_{k+1}^p(\phi) \xrightarrow{\partial_{k+1}^p(F,G)} U_k^{p+1}(\phi)$ *is an exact sequence if* $p + k > 0$.

(2) $U_k^{p+1}(\phi) \xrightarrow{\delta_k^{p+1}(F,G)} U_{k+1}^p(\phi) \xrightarrow{\delta_{k+1}^p(F,G)} U_{k+2}^{p-1}(\phi)$ *is exact if* $p \geqslant 1$.

Proof. To prove (1), observe that $U_{k+2}^{p-1}(\phi) = (\mathbb{A}^{p+k}(G) \otimes \mathbb{A}^{p+k}(F))_{k+1}$, $U_{k+1}^p(\phi) = (\mathbb{A}^{p+k}(G) \otimes \mathbb{A}^{p+k}(F))_k$, $U_k^{p+1}(\phi) = (\mathbb{A}^{p+k}(G) \otimes \mathbb{A}^{p+k}(F))_{k-1}$ and that the maps $\partial(F, G)$ are the boundary maps of the complex $\mathbb{A}^{p+k}(G) \otimes \mathbb{A}^{p+k}(F)$. Since the complexes $\mathbb{A}^{p+k}(F)$ (or $\mathbb{A}^{p+k}(G)$) are split exact for $p + k > 0$, the first part of this proposition is clearly true.

To prove (2), we make an observation similar to the one above. In this case, the modules in the sequence in (2) may be identified with $(\mathbb{B}_{p+k}(G) \otimes \mathbb{B}_{p+k}(F))_{k-1}$, $(\mathbb{B}_{p+k}(G) \otimes \mathbb{B}_{p+k}(F))_k$ and $(\mathbb{B}_{p+k}(G) \otimes \mathbb{B}_{p+k}(F))_{k+1}$, respectively. However, a careful look at the sign in the definition of $\delta_{k+1}^p(F, G)$ shows us that the maps of the modules in (2) are the boundary maps of the complex $\mathbb{B}_{p+k}(G) \otimes \overline{\mathbb{B}}_{p+k}(F)$ where by $\overline{\mathbb{B}}_{p+k}(F)$ we mean the complex whose terms are those of $\mathbb{B}_{p+k}(F)$ but whose boundary map is $-\delta_F$. Therefore, what we want to show is that $H^k(\mathbb{B}_{p+k}(G) \otimes \bar{\mathbb{B}}_{p+k}(F)) = 0$ if $p \geqslant 1$. Since the complexes $\mathbb{B}_{p+k}(G)$ and $\bar{\mathbb{B}}_{p+k}(F)$ are not exact, we need a lemma to complete our proof.

LEMMA 3.9. *Let*

$$\mathbb{C}: 0 \to C^0 \to C^1 \to \cdots,$$
$$\mathbb{D}: 0 \to D^0 \to D^1 \to \cdots$$

be free complexes. Suppose $t \leqslant s$ are integers such that

(a) $H^i(\mathbb{C}) = 0$ *for* $i \leqslant t$,

(b) $H^i(\mathbb{D}) = 0$ for $i \leqslant s$,

(c) $\operatorname{Coker}(C^t \to C^{t+1})$ is free.

Then $H^i(\mathbb{C} \otimes \mathbb{D}) = 0$ *for* $i \leqslant s + t + 1$.

Proof. Let $\bar{\mathbb{C}}$ be the complex

$$0 \to C^0 \to C^1 \to \cdots \to C^t \to C^{t+1} \to 0,$$

and let $L = \operatorname{Coker}(C^t \to C^{t+1})$. Then hypotheses (a) and (c) tell us that

$$0 \to C^0 \to C^1 \to \cdots \to C^t \to C^{t+1} \to L \to 0$$

is a split exact sequence.

A standard spectral sequence argument tells us that $H^i(\mathbb{C} \otimes \mathbb{D}) = H^i(\bar{\mathbb{C}} \otimes \mathbb{D})$ for $i \leqslant s + t + 1$, and another one tells us that $H^i(\bar{\mathbb{C}} \otimes \mathbb{D}) = H^{i-t-1}(L \otimes \mathbb{D})$. Since L is free, $H^{i-t-1}(L \otimes \mathbb{D}) = L \otimes H^{i-t-1}(\mathbb{D}) = 0$ if $i - t - 1 \leqslant s$ (by (b)). Hence $H^i(\mathbb{C} \otimes \mathbb{D}) = 0$ for $i \leqslant s + t + 1$.

We now apply this lemma to complete the proof of Proposition 3.8. If we let $\mathbb{C} = \mathbb{B}_{p+k}(G)$ and $\mathbb{D} = \bar{\mathbb{B}}_{p+k}(F)$, then (by Proposition 3.5) $\mathbb{C}$ and $\mathbb{D}$ satisfy the conditions of Lemma 3.9 for $s = t = [(p + k - 1)/2]$. Thus $H^i(\mathbb{B}_{p+k}(G) \otimes \mathbb{B}_{p+k}(F)) = 0$ for $i \leqslant 2[(p + k - 1)/2] + 1$. If $p \geqslant 1$, $k \leqslant 2[(p + k - 1)/2] + 1$ so that $H^k(\mathbb{B}_{p+k}(G) \otimes \bar{\mathbb{B}}_{p+k}(F)) = 0$ for $p \geqslant 1$, and (2) is proven.

DEFINITION 3.10. *For $p \geqslant 1$ and $k \geqslant 0$ define*

$$Z^p_{k+1}(F, G) = \operatorname{Coker}(U^{p+1}_k(\phi) \xrightarrow{\delta^{p+1}_k(F,G)} U^p_{k+1}(\phi)).$$

PROPOSITION 3.11. *The sequence*

$$0 \to Z^{p+1}_k(F, G) \to U^p_{k+1}(\phi) \to Z^p_{k+1}(F, G) \to 0 \tag{*}$$

is exact for $p \geqslant 1$. In addition, $Z^p_{k+1}(F, G)$ is universally free for $p \geqslant 2$.

Proof. The exactness of (*) is a trivial consequence of Proposition 3.8(2). To see that $Z_{k+1}^p(F, G)$ is universally free, it suffices to prove that $Z_{k+1}^p(F, G)$ is universally free, it suffices to prove that $Z_{k+1}^p(F, G)$ is free (for $p \geqslant 2$) when R is the ring of integers. For then, being the cokernel of a universal map of universally free modules, it will be universally free. But since $Z_{k+1}^p(F, G)$ is a submodule of $U_{k+2}^{p-1}(\phi)$, and since $U_{k+2}^{p-1}(\phi)$ is free, $Z_{k+1}^p(F, G)$ is a free R-module when R is the ring of integers.

LEMMA 3.12. *The diagram*

$$\begin{array}{ccc} U_k^{p+1}(\phi) & \xrightarrow{\partial(F,G)} & U_{k-1}^{p+2}(\phi) \\ {\scriptstyle \delta(F,G)}\big\downarrow & & \big\downarrow{\scriptstyle \delta(F,G)} \\ U_{k+1}^{p}(\phi) & \xrightarrow{\partial(F,G)} & U_k^{p+1}(\phi) \end{array}$$

is anticommutative.

Proof. Let $X \in \Lambda^{p+b}G \otimes D_a G$ and $y \in \Lambda^{p+a}F \otimes D_b F$ with $a + b = k - 1$. Then $x \otimes y$ (modulo rearrangement of terms) is a generator of $U_k^{p+1}(\phi)$. From the definitions of the maps $\partial(F, G)$ and $\delta(F, G)$ we get

$$\begin{aligned} \partial(F, G)\, \delta(F, G)(x \otimes y) = {}& \partial_G \delta_G(x) \otimes y + (-1)^{a+1}\, \delta_G(x) \otimes \partial_F(y) \\ & + (-1)^{a+1}\, \partial_G(x) \otimes \delta_F(y) - x \otimes \partial_F \delta_F(y); \\ \delta(F, G)\, \partial(F, G)(x \otimes y) = {}& \delta_G \partial_G(x) \otimes y + (-1)^{a}\, \partial_G(x) \otimes \delta_F(y) \\ & + (-1)^{a}\, \delta_G(x) \otimes \partial_F(y) - x \otimes \delta_F \partial_F(y). \end{aligned}$$

Therefore the sum of these two terms is

$$\begin{aligned} & (\partial_G \delta_G + \delta_G \partial_G)(x) \otimes y - x \otimes (\partial_F \delta_F + \delta_F \partial_F)(y) \\ & \quad = (p + k)x \otimes y - (p + k)x \otimes y = 0. \end{aligned}$$

By the above lemma we see that the maps

$$\partial_{k+1}^p(F, G) \colon U_{k+1}^p(\phi) \to U_k^{p+1}(\phi)$$

induce unique maps, which we will denote by $\bar{\partial}_{k+1}^p(F, G)$:

$$\bar{\partial}_{k+1}^p(F, G) \colon Z_{k+1}^p(F, G) \to Z_k^{p+1}(F, G).$$

DEFINITION 3.13. We define $X_{k+1}^p(1, F, G)$ to be the kernel of the map $\bar{\partial}_{k+1}^p(F, G)$.

PROPOSITION 3.14. *For $p \geqslant 1$, the sequence*

$$Z_{k+2}^{p-1}(F, G) \to Z_{k+1}^{p}(F, G) \to Z_{k}^{p+1}(F, G)$$

is exact. We therefore have an exact sequence

$$0 \to X_{k+1}^{p}(1, F, G) \to Z_{k+1}^{p}(F, G) \to X_{k}^{p+1}(1, F, G) \to 0 \qquad (**)$$

for all $p \geqslant 0$. It follows from this that $X_{k+1}^{p}(1, F, G)$ is universally free for $p \geqslant 2$.

Proof. By the exactness of sequence (*) above, we have a short exact sequence of complexes:

$$\begin{tikzcd}
\arrow[d] & \arrow[d] & \arrow[d] & \\
0 \arrow[r] & \mathrm{Ker} \arrow[r] \arrow[d] & U_{k+2}^{p-1}(\phi) \arrow[r] \arrow[d, "\partial(F,G)"] & Z_{k+2}^{p-1}(F,G) \arrow[r] \arrow[d, "\partial(F,G)"] & 0 \\
0 \arrow[r] & Z_{k}^{p+1}(F,G) \arrow[r] \arrow[d, "\partial(F,G)"] & U_{k+1}^{p}(\phi) \arrow[r] \arrow[d, "\partial(F,G)"] & Z_{k+1}^{p}(F,G) \arrow[r] \arrow[d, "\partial(F,G)"] & 0 \\
0 \arrow[r] & Z_{k-1}^{p+2}(F,G) \arrow[r] \arrow[d] & U_{k}^{p+1}(\phi) \arrow[r] \arrow[d] & Z_{k}^{p+1}(F,G) \arrow[r] \arrow[d] & 0 \\
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & Z_{1}^{p+k}(F,G) \arrow[r] \arrow[d] & U_{2}^{p+k-1}(\phi) \arrow[r] \arrow[d] & Z_{2}^{p+k-1}(F,G) \arrow[r] \arrow[d] & 0 \\
 & 0 \arrow[r] & U_{1}^{p+k}(\phi) \arrow[r] \arrow[d] & Z_{1}^{p+k}(F,G) \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

If we denote by H_{k+1}^{p} the homology of

$$Z_{k+2}^{p-1}(F, G) \to Z_{k+1}^{p}(F, G) \to Z_{k}^{p+1}(F, G),$$

and observe that the middle column of the above diagram is exact by Proposition 3.8 (1), we obtain

$$H_{k+1}^{p} = H_{k-1}^{p+2} = \cdots = H_{k+1-2i}^{p+2i} = 0 \qquad \text{for } i \text{ large.}$$

The exactness of (**) now follows immediately, and an argument similar to the proof of universal freeness of $Z_{k+1}^{p}(F, G)$ gives us the universal freeness of $X_{k+1}^{p}(1, F, G)$ for $p \geqslant 1$.

LEMMA 3.15. *The following diagrams are commutative*:

(i)
$$\begin{array}{ccc} U^p_{k+1}(\phi) & \xrightarrow{\partial^p_{k+1}(F,G)} & U^{p+1}_k(\phi) \\ \downarrow{\scriptstyle \partial^u_\phi} & & \downarrow{\scriptstyle \partial^u_\phi} \\ U^p_k(\phi) & \xrightarrow{\partial^p_k(F,G)} & U^{p+1}_{k-1}(\phi) \end{array}$$

(ii)
$$\begin{array}{ccc} U^p_{k+1}(\phi) & \xrightarrow{\delta^p_{k+1}(F,G)} & U^{p-1}_{k+2}(\phi) \\ \downarrow{\scriptstyle \partial^u_\phi} & & \downarrow{\scriptstyle \partial^u_\phi} \\ U^p_k(\phi) & \xrightarrow{\delta^p_k(F,G)} & U^{p-1}_{k+1}(\phi). \end{array}$$

Proof: Trivial.

The commutativity of diagram (ii) above induces unique maps $\partial^Z_\phi : Z^p_{k+1}(F, G) \to Z^p_k(F, G)$ with $\partial^Z_\phi \partial^Z_\phi = 0$.

DEFINITION 3.16. The complex $\{Z^p_{k+1}(F, G), \partial^Z_\phi\}$ will be denoted by $\mathbb{Z}^p(\phi)$.

Lemmas 3.12 and 3.15(i) show us that the map $\bar{\partial}(F, G)$: $\mathbb{Z}^p(\phi) \to \mathbb{Z}^{p+1}(\phi)$ (sending $Z^p_{k+1}(F, G)$ to $Z^{p+1}_k(F, G)$) is a map of complexes, and thus the map $\partial^Z_\phi : Z^p_{k+1}(F, G) \to Z^p_k(F, G)$ induces a map $\partial^X_\phi : X^p_{k+1}(1, F, G) \to X^p_k(1, F, G)$.

DEFINITION 3.17. The complex $\{X^p_{k+1}(1, F, G), \partial^X_\phi\}$ will be denoted by $\mathbb{X}^p(1, \phi)$.

Clearly, the complex $\mathbb{X}^p(1, \phi)$ is the kernel of the map of complexes

$$\bar{\partial}(F, G): \mathbb{Z}^p(\phi) \to \mathbb{Z}^{p+1}(\phi).$$

Remark 3.18. If R_0 is a commutative ring, and F_0, G_0 are free R_0-modules, let $R = S(F_0 \otimes G_0) =$ the symmetric algebra over R_0 of $F_0 \otimes G_0$. Then R is a graded ring and $F = R \otimes F_0$, $G = R \otimes G_0$ are free R-modules. A morphism $F_0 \otimes G_0 \to R$ induces a morphism of $F \otimes G$ into R, and thus the inclusion $F_0 \otimes G_0 \to S(F_0 \otimes G_0)$, which is the identity from $F_0 \otimes G_0 \to R_1 = F_0 \otimes G_0$, gives us the generic map $\phi: F \otimes G \to R$. From the universal freeness of the functors $Z^p_{k+1}(F, G)$, and $X^p_{k+1}(1, F, G)$, it is clear that $Z^p_{k+1}(F, G) = R \otimes Z^p_{k+1}(F_0, G_0)$ and $X^p_{k+1}(1, F, G) = R \otimes X^p_{k+1}(1, F_0, G_0)$. Tracing through the definitions of the maps ∂^u_ϕ, ∂^z_ϕ and ∂^x_ϕ, and using the fact that the generic map ϕ is the map induced by the identity map (and is, therefore, of degree 1), it is straightforward to see that the maps ∂_ϕ^z and ∂_ϕ^x are induced from maps

$$Z_{k+1}^p(F_0, G_0) \to Z_k^p(F_0, G_0) \otimes R_1,$$
$$X_{k+1}^p(1, F, G_0) \to X_k^p(1, F_0, G_0) \otimes R_1.$$

Thus the complexes $\mathbb{Z}^p(\phi)$ and $\mathbb{X}^p(1, \phi)$ are linear complexes as in Definition 2.6.

Remark 3.19. In [4], we defined $X_{k+1}^p(F, G)$ to be the kernel of the map

$$\Lambda^k(F \otimes G) \otimes \Lambda^p F \otimes \Lambda^p G \to \Lambda^{k-1}(F \otimes G) \otimes S_{p+1}(F \otimes G)$$

(the above map is the composition

$$\Lambda^k(F \otimes G) \otimes \Lambda^p F \otimes \Lambda^p G \xrightarrow{1 \otimes \alpha} \Lambda^k(F \otimes G) \otimes S_p(F \otimes G)$$
$$\xrightarrow{\partial} \Lambda^{k-1}(F \otimes G) \otimes S_{p+1}(F \otimes G)$$

where α is the canonical embedding $\Lambda^p F \otimes \Lambda^p G \to S_p(F \otimes G)$ and ∂ is the Koszul map, that is, it is the differential of the Koszul complex $\Lambda(F \otimes G) \otimes S(F \otimes G)$).

The behavior of the modules $X_{k+1}^p(F, G)$ under decomposition led us to introduce the modules $Z_{k+1}^{s,t}(F, G)$ defined as the kernel of the map

$$\Lambda^k(F \otimes G) \otimes \Lambda^s F \otimes \Lambda^t G \to \Lambda^{k-1}(F \otimes G) \otimes L_2^s F \otimes L_2^t G.$$

It follows immediately from the above definitions that we have exact sequences

$$0 \to X_{k+1}^p(F, G) \to X_k^p(F, G) \otimes S_1(F \otimes G) \to X_{k-1}^p(F, G) \otimes S_2(F \otimes G),$$
$$0 \to Z_{k+1}^{s,t}(F, G) \to Z_k^{s,t}(F, G) \otimes S_1(F \otimes G) \to Z_{k-1}^{s,t}(F, G) \otimes S_2(F \otimes G),$$

which means that we get linearly exact complexes

$$\cdots \to X_{k+1}^p(F, G) \otimes S(F \otimes G) \to X_k^p(F, G) \otimes S(F \otimes G) \to$$
$$\cdots \to Z_{k+1}^{s,t}(F, G) \otimes S(F \otimes G) \to Z_k^{s,t}(F, G) \otimes S(F \otimes G) \to \cdots$$

of degree 1 over the polynomial ring $S(F \otimes G)$.

DEFINITION. $I_p(F \otimes G)$ is the homogeneous ideal of the graded ring $S(F \otimes G)$ generated by the image of the canonical embedding $\Lambda^p F \otimes \Lambda^p G \to S_p(F \otimes G)$. We write $(I_p(F \otimes G))_k$ for $I_p(F \otimes G) \cap S_k(F \otimes G)$.

Observing that $(I_p(F \otimes G))_p = \Lambda^p F \otimes \Lambda^p G$ and $(I_p(F \otimes G))_{p+1}/(I_{p+1}(F \otimes G))_{p+1} = L_2^p F \otimes L_2^p G$, we see that $X_{k+1}^p(F, G)$ is the kernel of the map

$$\Lambda^k(F \otimes G \otimes (I_p(F \otimes G))_p \xrightarrow{\partial} \Lambda^{k-1}(F \otimes G) \otimes (I_p(F \otimes G))_{p+1}$$

and that $Z_{k+1}^{p,p}(F, G)$ is the kernel of the map

$$\Lambda^k(F \otimes G) \otimes (I_p(F \otimes G))_p \xrightarrow{\partial} \Lambda^{k-1}(F \otimes G) \otimes (I_p(F \otimes G))_{p+1}/(I_{p+1}(F \otimes G))_{p+1},$$

where both maps come from the Koszul differential. It follows that $X_{k+1}^p(F, G)$ is contained in $Z_{k+1}^{p,p}(F, G)$ and that there is an induced map $Z_{k+1}^{p,p}(F, G) \to X_k^{p+1}(F, G)$. Therefore one may define $X_{k+1}^p(F, G)$ as the kernel of the map $Z_{k+1}^{p,p}(F, G) \to Z_k^{p+1,p+1}(F, G)$ which is formally similar to the definition of $X_{k+1}^p(F, G)$ given in this paper. Moreover, using decomposition techniques similar to those in [4], one can prove that the sequence

$$\begin{aligned}0 &\to \Lambda^{p+k}(F \otimes G) \to Z_{p+k}^{1,1}(F, G) \to \cdots \to Z_{k+1}^{p,p}(F, G) \to Z_k^{p+1,p+1}(F, G)\\ &\to \cdots \to \Lambda^{p+k}F \otimes \Lambda^{p+k}G \to 0\end{aligned}$$

is exact, or equivalently that the sequence

$$0 \to X_{k+1}^p(F, G) \to Z_{k+1}^{p,p}(F, G) \to X_k^{p+1}(F, G) \to 0$$

is exact.

DEFINITION. We define a map $U_{k+1}^p(F, G) \to Z_k^{p,p}(F, G)$ to be the composition

$$\begin{array}{c}\sum_{a+b=k} \Lambda^{p+a}F \otimes D_a G \otimes \Lambda^{p+b}G \otimes D_b F\\ \downarrow\\ \sum \Lambda^p F \otimes \Lambda^a F \otimes D_a G \otimes \Lambda^p G \otimes \Lambda^b G \otimes D_b F\\ \downarrow\\ \Lambda^k(F \otimes G) \otimes \Lambda^p F \otimes \Lambda^p G.\end{array}$$

When R contains the field Q of rationals, it is easy to check that there is an exact sequence

$$U_k^{p+1}(F, G) \xrightarrow{\delta_k^{p+1}} U_{k+1}^p(F, G) \to Z_k^{p,p}(F, G) \to 0$$

when $p \geqslant 2$, so that $Z_k^{p,p}(F, G)$ is $Z_k^p(F, G)$ in this case. This is also true (but harder to prove) under the weaker assumption that R contains the ring $\mathbb{Z}$ of integers.

4. The Schur Complexes

In this section F, G are free R-modules of ranks m, n respectively and $\phi: G \to F$ is an R-map. We denote by c_ϕ the element of $F \otimes G^*$ corresponding to ϕ under the canonical isomorphism $\mathrm{Hom}_R(G, F) \cong F \otimes G^*$. More detailed discussion of the material in this section will be found in [2].

Definition 4.1. (a) The symmetric algebra $S\phi$ of the morphism ϕ is the R-bialgebra $SF \otimes \Lambda G$ formed by taking the usual tensor product of the R-bialgebras SF and ΛG. We let $M_{S\phi}: S\phi \otimes S\phi \to S\phi$, $\Delta_{S\phi}: S\phi \to S\phi \otimes S\phi$, and $T_{S\phi}: S\phi \otimes S\phi \to S\phi \otimes S\phi$ denote the multiplication, the comultiplication, and the commutation map respectively of the R-bialgebra $S\phi$.

(b) We put a complex structure on $S\phi$ as follows: let $(S\phi)_j = \sum_{i=0}^{\infty} S_i F \otimes \Lambda^j G$ be the jth degree of the complex and let $\partial_{S\phi}: (S\phi)_j \to (S\phi)_{j-1}$ be the R-map given by the action of $c_\phi \in SF \otimes \Lambda G^*$ on $SF \otimes \Lambda G$. It is easy to check that this makes $S\phi$ into a complex.

Lemma 4.2. $M_{S\phi}$, $\Delta_{S\phi}$, $T_{S\phi}$ are all "compatible" with the differential $\partial_{S\phi}$ (for example, $M_{S\phi}$ is "compatible" with $\partial_{S\phi}$ means that the following diagram is commutative:

$$\begin{tikzcd} S\phi \otimes S\phi \arrow[r, "M_{S\phi}"] \arrow[d, "\partial_{S\phi \otimes S\phi}"'] & S\phi \arrow[d, "\partial_{S\phi}"] \\ S\phi \otimes S\phi \arrow[r, "M_{S\phi}"] & S\phi \end{tikzcd}$$

where $\partial_{S\phi \otimes S\phi}$ is the differential of the tensor product of complexes).

Definition 4.3. $S_k\phi$ is the subcomplex of $S\phi$ given by $0 \to \Lambda^k G \to F \otimes \Lambda^{k-1} G \to \cdots \to S_{k-j} F \otimes \Lambda^j G \to \cdots \to S_k F \to 0$ where the jth degree component $(S_k\phi)_j = S_{k-j} = S_{k-j} F \otimes \Lambda^j G$. Note that $S_0\phi$ is the complex $0 \to R \to 0$ with $(S_0\phi)_0 = R$.

Definition 4.4. (a) The exterior algebra $\Lambda\phi$ of the morphism ϕ is the R-bialgebra $\Lambda F \dot{\otimes} DG$ formed by taking the antisymmetric tensor product (denoted $\dot{\otimes}$) of the R-bialgebras ΛF and DG. We let $m_{\Lambda\phi}: \Lambda\phi \otimes \Lambda\phi \to \Lambda\phi$, $\Delta_{\Lambda\phi}: \Lambda\phi \to \Lambda\phi \otimes \Lambda\phi$, and $T_{\Lambda\phi}: \Lambda\phi \otimes \Lambda\phi \to \Lambda\phi \otimes \Lambda\phi$ denote the multiplication, the comultiplication, and the commutation map respectively of the R-bialgebra $\Lambda\phi$.

(b) We put a complex structure on $\Lambda\phi$ as follows: let $(\Lambda\phi)_j = \sum_i \Lambda^i F \otimes D_j G$ be the jth degree component of the complex and $\partial_{\Lambda\phi}: (\Lambda\phi)_j \to (\Lambda\phi)_{j-1}$ be the R-map given by the action of $C\phi \in \Lambda F \otimes SG^*$ on $\Lambda F \dot{\otimes} DG$.

Remark 4.5. Here we give an explanation of what is meant by the R-bialgebra structure of the antisymmetric tensor product $\Lambda F \dot{\otimes} DG$. If $M = \sum M_i$, $N = \sum N_j$ are graded R-modules, we define the antisymmetric twisting morphism $\dot{T}: M \otimes N \to N \otimes M$ to be the R-map given as follows: if $x \otimes y \in M_i \otimes N_j$, then $\dot{T}(x \otimes y) = (-1)^{ij} y \otimes x$. Then $m_{\Lambda\phi}: \Lambda\phi \otimes \Lambda\phi \to \Lambda\phi$ is the composition

$$\Lambda F \otimes DG \otimes \Lambda F \otimes DG \xrightarrow{1 \otimes \dot{T} \otimes 1} \Lambda F \otimes \Lambda F \otimes DG \otimes DG$$
$$\xrightarrow{m_{\Lambda F} \otimes m_{DG}} \Lambda F \otimes DG,$$

$\Delta_{\Lambda\phi}: \Lambda\phi \to \Lambda\phi \otimes \Lambda\phi$ is the composition

$$\Lambda F \otimes DG \xrightarrow{\Delta_{\Lambda F} \otimes \Delta_{DG}} \Lambda F \otimes \Lambda F \otimes DG \otimes DG$$
$$\xrightarrow{1 \otimes \dot{T} \otimes 1} \Lambda F \otimes DG \otimes \Lambda F \otimes DG$$

and $T_{\Lambda\phi}: \Lambda\phi \otimes \Lambda\phi \to \Lambda\phi \otimes \Lambda\phi$ is the map given by if

$$x_1 \otimes y_1 \otimes x_2 \otimes y_2 \in \Lambda^{a_1} F \otimes D_{b_1} G \otimes \Lambda^{a_2} F \otimes D_{b_2} G,$$

$$T_{\Lambda\phi}(x_1 \otimes y_1 \otimes x_2 \otimes y_2)$$
$$= (-1)^{a_1 a_2 + a_1 b_2 + a_2 b_1} x_2 \otimes y_2 \otimes x_1 \otimes y_1 \in \Lambda^{a_2} F \otimes D_{b_2} G \otimes \Lambda^{a_1 F} \otimes D_{b_1} F.$$

Also the action of $\Lambda F \dot{\otimes} SG^*$ on $\Lambda F \dot{\otimes} DG$ is the composition $\Lambda F \otimes SG^* \Lambda F \otimes DG^*$ on $\Lambda F \dot{\otimes} DG$ is the composition

$$\Lambda F \otimes SG^* \otimes \Lambda F \otimes DG \xrightarrow{1 \otimes \dot{T} \otimes 1} \Lambda F \otimes \Lambda F \otimes SG^* \otimes DG$$
$$\xrightarrow{m_{\Lambda F} \otimes n_{SG^*}} \Lambda F \otimes DG$$

where $n_{SG^*}: SG^* \otimes DG \to DG$ is the natural action of SG^* on DG.

LEMMA 4.6. *$m_{\Lambda\phi}, \Delta_{\Lambda\phi}, T_{\Lambda\phi}$ are all "compatible" with the differential $\partial_{\Lambda\phi}$.*

DEFINITION 4.7. $\Lambda^k\phi$ is the subcomplex of $\Lambda\phi$ given by $0 \to D_k G \to F \otimes D_{k-1} G \to \cdots \to \Lambda^{k-j} F \otimes_j G \to \cdots \to \Lambda^k F \to 0$ where the jth degree component $(\Lambda^k\phi)_j = \Lambda^{k-j} F \otimes D_j G$. Note that $\Lambda^0\phi$ is the complex $0 \to R \to 0$ with $(\Lambda^0\phi)_0 = R$.

Remark 4.8. It is easy to see that $\Lambda\phi = \sum_k \Lambda^k\phi$, $S\phi = \sum_k S_k\phi$ as direct sums of complexes.

DEFINITION 4.9. Let $\lambda = (\lambda_1,..., \lambda_s)$ $\mu = (\mu_1,..., \mu_s)$ be partitions such that $\mu \subseteq \lambda$. We define $\Lambda_{\lambda/\mu}\phi$ to be the tensor product of complexes $\Lambda^{\lambda_1-\mu_1}\phi \otimes \cdots \otimes \Lambda^{\lambda_s-\mu_s}\phi$ and $S_{\lambda/\mu}\phi$ to be the tensor product of complexes $S_{\lambda_1-\mu_1}\phi \otimes \cdots \otimes S_{\lambda_s-\mu_s}\phi$.

Now we are ready to define the Schur complex on ϕ.

DEFINITION 4.10. Let $\lambda = (\lambda_1,..., \lambda_s)$, $\mu = (\mu_1,..., \mu_s)$ be partitions such that $\mu \subseteq \lambda$. Let $\tilde{\lambda} = (\tilde{\lambda}_1,..., \tilde{\lambda}_t)$, $\tilde{\mu} = (\tilde{\mu}_1,..., \tilde{\mu}_t)$ be their transposes. Let (α_{ij}) be the $s \times t$ matrix defined by

$$\begin{aligned} \alpha_{ij} &= 1 \qquad \text{if} \quad \mu_i + 1 \leqslant j \leqslant \lambda_i, \\ &= 0 \qquad \text{otherwise.} \end{aligned}$$

We define the Schur map $d_{\lambda/\mu}(\phi): \Lambda_{\lambda/\mu}(\phi) \to S_{\tilde{\lambda}/\tilde{\mu}}$ to be the composition

$$\begin{array}{c} \Lambda^{\lambda_1-\mu_1}\phi \otimes \cdots \otimes \Lambda^{\lambda_s-\mu_s}\phi \\ \downarrow \\ (\Lambda^{\alpha_{11}}\phi \otimes \cdots \otimes \Lambda^{\alpha_{1t}}\phi) \otimes \cdots \otimes (\Lambda^{\alpha_{s1}}\phi \otimes \cdots \otimes \Lambda^{\alpha_{st}}\phi) \\ \downarrow \\ (S_{\alpha_{11}}\phi \otimes \cdots \otimes S_{\alpha_{1t}}\phi) \otimes \cdots \otimes (S_{\alpha_{s1}}\phi \otimes \cdots \otimes S_{\alpha_{st}}\phi) \\ \downarrow \\ S_{\tilde{\lambda}_1-\tilde{\mu}_1}\phi \otimes \cdots \otimes S_{\tilde{\lambda}_t-\tilde{\mu}_t}\phi, \end{array}$$

where the first map is the tensor product of the maps $\Delta_{\Lambda\phi}: \Lambda^{\lambda_i-\mu_i}\phi \to \Lambda^{\alpha_{i1}}\phi \otimes \cdots \otimes \Lambda^{\alpha_{it}}\phi$ $(i = 1,..., s)$, the second map is the tensor product of the canonical isomorphisms $\Lambda^{\alpha_{ij}}\phi \to S_{\alpha_{ij}}\phi$ (remember that $\alpha_{ij} = 0$ or 1), and the third map is multiplication in the algebra $\otimes^t S\phi = S\phi \otimes \cdots \otimes S\phi$. It follows from Lemmas 4.2 and 4.6 that the first and third maps are maps of complexes, and it is trivial to check that the middle map is a map of complexes. Therefore $d_{\lambda/\mu}(\phi)$ is a complex map. The image of $d_{\lambda/\mu}(\phi): \Lambda_{\lambda/\mu}(\phi) \to S_{\tilde{\lambda}/\tilde{\mu}}(\phi)$ is called the Schur complex on the morphism ϕ and is denoted $L_{\lambda/\mu}\phi$.

We give another definition of the Schur complex:

DEFINITION 4.11. Define the map of complexes $\square_{\Lambda\phi}: \Lambda\phi \otimes \Lambda\phi \to \Lambda\phi \otimes \Lambda\phi$ to be the composition

$$\Lambda\phi \otimes \Lambda\phi \xrightarrow{\Delta_{\Lambda\phi}\otimes 1} \Lambda\phi \otimes \Lambda\phi \otimes \Lambda\phi \xrightarrow{1\otimes m_{\Lambda\phi}} \Lambda\phi \otimes \Lambda\phi.$$

We define $\bar{L}_{\lambda/\mu}\phi$ to be the cokernel of the map of complexes

$$\begin{array}{c} \sum_{k=1}^{s-1} \sum_{i=\mu_k-\mu_{k+1}+1}^{\lambda_{k+1}-\mu_{k+1}} \Lambda^{\lambda_1-\mu_1}\phi \otimes \cdots \otimes \Lambda^{\lambda_k-\mu_k+i}\phi \otimes \Lambda^{\lambda_{k+1}-\mu_{k+1}-i}\phi \otimes \cdots \otimes \Lambda^{\lambda_s-\mu_s}\phi \\ \Big\downarrow \sum_{k=1}^{s-1} 1 \otimes \cdots \otimes \square^k \otimes \cdots \otimes 1 \\ \Lambda^{\lambda_1-\mu_1}\phi \otimes \cdots \otimes \Lambda^{\lambda_s-\mu_s}\phi, \end{array}$$

where $\square^k$ is the map

$$\square_{\Lambda\phi}: \sum_{i=\mu_k-\mu_{k+1}+1}^{\lambda_{k+1}-\mu_{k+1}} \Lambda^{\lambda_k-\mu_k+i}\phi \otimes \Lambda^{\lambda_{k+1}-\mu_{k+1}-i}\phi \to \Lambda^{\lambda_k-\mu_k}\phi \otimes \Lambda^{\lambda_{k+1}-\mu_{k+1}}\phi.$$

The main theorem or Schur Complexes is

THEOREM 4.12. (1) *There is a natural isomorphism* $\bar{L}_{\lambda/\mu}(\phi) \to L_{\lambda/\mu}(\phi)$ *of complexes (consequently,* $L_{\lambda/\mu}(\phi)$ *is universally free).*

(2) *If* $\phi = \phi_1 \oplus \phi_2$, *then there is a natural filtration of* $L_{\lambda/\mu}$ *by complexes whose associated graded object is* $\sum_{\mu\subseteq\sigma\subseteq\lambda} L_{\sigma/\mu}(\phi_1) \otimes L_{\lambda/\sigma}(\phi_2)$.

COROLLARY 4.13. *Let* $(L_{\lambda/\mu}(\phi))_j =$ the jth *degree component of* $L_{\lambda/\mu}\phi$. *There is a natural filtration on* $(L_{\lambda/\mu}(\phi))_j$ *whose associated graded object is* $\sum_{\mu\subseteq\sigma\subseteq\lambda,(\bar{\sigma})=j} L_{\sigma/\mu}(F) \otimes K_{\lambda/\sigma}(G)$.

Proof. Observe that $(L_{\lambda/\mu}(\phi))_j$ depends only on the modules F, G and not on the map ϕ. Therefore, if we take $\phi_1: 0 \to F$, $\phi_2: G \to 0$ we have $L_{\lambda/\mu}(\phi)_j = L_{\lambda/\mu}(\phi_1 + \phi_2)_j$. By the theorem, there is a natural filtration of $(L_{\lambda/\mu}(\phi_1 \oplus \phi_2))_j$ whose associated graded object is $\sum_{\mu<\sigma<\lambda} (L_{\sigma/\mu}(\phi_1) \otimes L_{\lambda/\sigma}(\phi_2))_j$. Noting that $(L_{\sigma/\mu}(\phi_1))_i = 0$ if $i \neq 0$, we have $(L_{\sigma/\mu}(\phi_1) \otimes L_{\lambda/\sigma}(\phi_2))_j = (L_{\sigma/\mu}(\phi_{10} \otimes (L_{\lambda/\sigma}(\phi))_j = (L_{\sigma/\mu}(F) \otimes (K_{\lambda/\sigma}(G))$.

COROLLARY 4.14. *Let* $\phi: G \to F$ *be a split injection. Then* $L_{\lambda/\mu}(\phi)$ *is acyclic and* $H_0(L_{\lambda/\mu}(\phi)) = L_{\lambda/\mu}(\mathrm{Coker}\,\phi)$.

Proof. We proceed by induction on rank G, the case $G = 0$ being trivial. If rank $G > 0$, we can split $\phi: G \to F$ into a direct sum $\phi_1 \oplus \phi_2: R \oplus G^1 \to R \oplus F^1$ where $\phi_1: R \to R$ is the identity 1_R and $\phi_2: G^1 \to F^1$ is a split injection. By the theorem, $L_{\lambda/\mu}(\phi)$ decomposes into $\sum_{\mu\subseteq\sigma\subseteq\lambda} L_{\sigma/\mu}(\phi_1) \otimes L_{\lambda/\sigma}(\phi_2)$ up to filtration. It is easy to check that $L_{\sigma/\mu}(R \to^{1_R} R)$ is exact for $\sigma \neq \mu$. It follows that $L_{\lambda/\mu}(\phi)$ is homotopically equivalent to $L_{\mu/\mu}(\phi_1) \otimes L_{\lambda/\mu}(\phi_2)$. But $L_{\mu/\mu}(\phi_1)$ is the complex $0 \to R \to 0$ with R in degree 0. Therefore $L_{\lambda/\mu}(\phi)$ is homotopically equivalent to $L_{\lambda/\mu}(\phi_2)$. By induction, we are done

5. The Resolution of the Submaximal Minors

Let F and G be free R-modules of ranks m and p, where $m \geqslant p$, respectively, and let (p^r) denote the partition

$$\underbrace{(p,..., p)}_{r}.$$

As in Section 3, we let I_p denote the ideal in $S(F \otimes G)$ generated by $\Lambda^p F \otimes \Lambda^p G$ (i.e., the ideal generated by the minors of order p of the generic map), and I_p^r denotes the rth power of I_p. Because $p = \operatorname{rank} G$, we have a canonical injection of the tensor product of Schur functors $L_{(p^r)}F \otimes L_{(p^r)}G \to I_p^r$ (by the standard basis theorem [1, 8, 9]). Consequently, we have a map

$$\eta_k(r, F, G): \Lambda^k(F \otimes G) \otimes L_{(p^r)}F \otimes L_{(p^r)}G \to \Lambda^{k-1}(F \otimes G) \otimes I_p^r$$

which is the composition of the maps

$$\Lambda^k(F \otimes G) \otimes L_{(p^r)}F \otimes L_{(p^r)}G$$
$$\xrightarrow{\alpha} \Lambda^{k-1}(F \otimes G) \otimes (F \otimes G) \otimes L_{(p^r)}F \otimes L_{(p^r)}G \xrightarrow{\beta} \Lambda^{k-1}(F \otimes G) \otimes I_p^r,$$

where α is obtained by diagonalizing $\Lambda^k(F \otimes G)$ and β is obtained by multiplying the image of $L_{(p^r)}F \otimes L_{(p^r)}G$ in I_p^r by $F \otimes G$ in $S(F \otimes G)$.

Definition 5.1. For $k \geqslant 0$, we denote the kernel of the map $\eta_k(r, F, G)$ by $Y_{k+1}(r, F, G)$.

Proposition 5.2. (1) $Y_1(r, F, G) = L_{(p^r)}F \otimes L_{(p^r)}G$.

(2) $0 \to Y_2(r, F, G) \to Y_1(r, F, G) \otimes F \otimes G \to I_p^r$ is exact.

(3) For each $k \geqslant 2$ there is an exact sequence

$$0 \to Y_{k+1}(r, F, G) \to Y_k(r, F, G) \otimes (F \otimes G) \to Y_{k-1}(r, F, G) \otimes S_2(F \otimes G).$$

Proof. (1) and (2) are simply restatements of the definition. To obtain the map

$$\partial^Y: Y_{k+1}(r, F, G) \to Y_k(r, F, G) \otimes (F \otimes G)$$

we observe that the diagonal map

$$\Lambda^k(F \otimes G) \to \Lambda^{k-1}(F \otimes G) \otimes (F \otimes G)$$

tensored with the identity on $L_{(p^r)}F \otimes L_{(p^r)}G$ induces a unique map from $Y_{k+1}(r, F, G)$ to $Y_k(r, F, G)$. The map $Y_k(r, F, G) \otimes (F \otimes G) \to Y_{k-1}(r, F, G) \otimes S_2(F \otimes G)$ is just the composition

$$\begin{aligned} Y_k(r, F, G) \otimes (F \otimes G) &\xrightarrow{\partial^Y \otimes 1} Y_{k-1}(r, F, G) \otimes (F \otimes G) \otimes (F \otimes G) \\ &\xrightarrow{1 \otimes \mu} Y_{k-1}(r, F, G) \otimes S_2(F \otimes G), \end{aligned}$$

where μ is multiplication in $S(F \otimes G)$.

The exactness claimed in (3) is a consequence of the acyclicity of the Koszul complex $\Lambda(F \otimes G) \otimes S(F \otimes G)$.

If $\phi: F \otimes G \to R$ is a map, then we obtain a map

$$\partial^Y_\phi: Y_{k+1}(r, F, G) \to Y_k r, F, G)$$

for $k \geqslant 1$ which is the composition

$$\begin{aligned} Y_{k+1}(r, F, G) &\xrightarrow{\partial^Y} Y_k(r, F, G) \otimes (F \otimes G) \\ &\xrightarrow{1 \otimes \phi} Y_k(r, F, G) \otimes R = Y_k(r, F, G). \end{aligned}$$

It is easy to check that $\partial^Y_\phi \circ \partial^Y_\phi = 0$ so we obtain a complex $\{Y_{k+1}(r, F, G), \partial^Y_\phi\}$.

DEFINITION 5.3. The complex $\{Y_{k+1}(r, F, G), \partial^Y_\phi\}$ will be denoted by $\curlyvee(r, \phi)$. When ϕ is the generic map, this complex will be denoted by $\curlyvee(r, F, G)$.

THEOREM 5.4. *Let $\phi: F \otimes G \to R$ be a map as above with G of rank p, and suppose that for each $j = 1,..., p$ the ideal $I_j(\phi)$ generated by the minors of ϕ of order j has grade $\geqslant (p + 1 - j)(m - p) + 1$. Then $\curlyvee(r, \phi)$ is a free resolution of $I^r_p(\phi)$ and this resolution can be augmented to give the resolution of $R/I^r_p(\phi)$.*

Proof. To prove this theorem, we completely ignore the complex $\curlyvee(r, \phi)$ for the moment and look instead at the Schur complex $L_{((m-p)^r)}(G \to^\phi F^*)$. We will show that this complex tensored with the module $L_{(p^r)}G$ resolves $I^r_p(\phi)$. If we show this, it will follow that $\curlyvee(r, \phi)$ is the same complex as $L_{((m-p)^r)}(G \to F^*) \otimes L_{(p^r)}G$ because this latter complex is linear and, being exact, is linearly exact. Since both complexes, $L_{((m-p)^r)}(G \to F^*) \otimes L_{(p^r)}G$ and $\curlyvee(r, \phi)$ have the same initial term, and since both are linearly exact when ϕ is the generic map, it follows that both complexes are the same.

Since G has rank p, $L_{(p^r)}G \approx R$, so we may dispense with writing the term $L_{(p^r)}G$ when proving the acyclicity of $L_{((m-p)^r)}(G \to F^*)$. By Corollary 4.13, we know that the length of the complex $L_{((m-p)^r)}(G \to F^*)$ is less than or equal to $(\min(r, p))(m-p)$. By the acyclicity lemma, it suffices to prove that this complex is acyclic if we localize at primes of localized grade $< \min(r, p)(m-p)$ (see [6] for a statement of the acyclicity lemma). Our assumption on grade $I_1(\phi)$ tells us that if we localize at such a prime, then $I_1(\phi)$ becomes R (where R now stands for the localization of R), and $\phi: G \to F^*$ equals $\phi^1 \oplus 1: G^1 \oplus R \to F^{1*} \oplus R$ where the grades of $I_j(\phi^1)$ satisfy the same hypotheses as those for $I_j(\phi)$ for $j = 1, \ldots, p-1$. Again, using Theorem 4.12 and Corollary 4.14, we see that $L_{((m-p)^r)}(G^1 \to F^{1*})$ is homotopically equivalent to $L_{((m-p))^r)}(G \to F^*)$ and induction on $\operatorname{rank} F + \operatorname{rank} G$ finishes the proof that $L_{(m-p)^r)}(G^1 \to F^{1*})$ is acyclic. This completes the proof of our theorem.

We are now ready to pass to the resolution of the $p \times p$ minors when $\operatorname{rank} G = p + 1$. We will assume that ϕ is the generic map and that F and G are fixed. Thus we will write $\mathbb{U}^p$, $\mathbb{Z}^p$ and $\mathbb{X}^p(1)$ for $\mathbb{U}^p(\phi)$, $\mathbb{Z}^p(\phi)$ and $\mathbb{X}^p(1, \phi)$, and will also write Z^p_{k+1}, $X^p_{k+1}(1)$ for $Z^p_{k+1}(F, G)$ and $X^p_{k+1}(1, F, G)$.

PROPOSITION 5.5.

(a) $H_i(\mathbb{X}^p(1)) = 0$ *for* $i > 3$.

(b) $H_i(\mathbb{X}^p(1)) \approx H_i(\mathbb{Z}^p)$ *for* $i \geqslant 2$.

(c) $H_i(\mathbb{U}^p) = 0$ *for* $i \geqslant 3$.

(d) $\mathbb{X}^{p+1}(1) \approx \mathbb{Z}^{p+1}$ *is a resolution of* $I_{p+1}(\phi)$.

(e) *We have the exact sequence*

$$0 \to H_3(\mathbb{Z}^p) \to I_{p+1}(\phi) \to H_2(\mathbb{U}^p) \to H_2(\mathbb{Z}^p) \to 0.$$

Proof. First we prove (d). But this is just the Eagon–Northcott complex for the maximal order minors (it also follows from Theorem 5.4 letting $r = 1$).

Next, we recall the two exact sequences of Section 3:

$$0 \to Z^{p+1}_k \to U^p_{k+1} \to Z^p_{k+1} \to 0, \tag{*}$$

$$0 \to X^p_{k+1}(1) \to Z^p_{k+1} \to X^{p+1}_k(1) \to 0. \tag{**}$$

We saw, in fact, that the above sequences are exact sequences of complexes

$$0 \to \mathbb{Z}^{p+1} \to \mathbb{U}^p \to \mathbb{Z}^p \to 0 \tag{*}$$

$$0 \to \mathbb{X}^1(1) \to \mathbb{Z}^p \to \mathbb{X}^{p+1} \to 0 \tag{**}$$

provided we keep in mind the dimension shifts.

To prove (c), we use the fact that $\mathbb{U}^p$ is he tensor product of two complexes:

$$(\cdots \to \Lambda^{p+1}F \otimes G \to \Lambda^p F) \otimes (\Lambda^{p+1}G \otimes F \to \Lambda^p G)$$

with the left-hand complex acyclic (see [3] for a proof of acyclicity of this complex). Let us call this acyclic complex $\mathbb{A}$. Then $\mathbb{U}^p$ is the mapping cone of $\mathbb{A} \otimes \Lambda^{p+1}G \otimes F \to \mathbb{A} \otimes \Lambda^p G$ and the acyclicity of $\mathbb{A}$ gives us (c). (Do not forget that the indexing of our complex $\mathbb{U}^p$ starts with 1, not 0, which accounts for the condition $i \geqslant 3$ instead of $i \geqslant 2$.)

To prove (b) we use the sequence (**). From the associated homology sequence (and the acyclicity of $\mathbb{X}^{p+1}(1)$) we get

$$H_i(\mathbb{X}^p(1)) = H_i(\mathbb{Z}^p) \qquad \text{for} \quad i \geqslant 3$$

and

$$0 \to H_2(\mathbb{X}^p(1)) \to H_2(\mathbb{Z}^p) \to I_{p+1}(\phi) \to I_p(\phi) \to I_p(\phi)/I_{p+1}(\phi) \to 0,$$

which shows that $H_2(\mathbb{X}^p(1)) = H_2(\mathbb{Z}^p)$.

To prove (a) it suffices to prove that $H_i(\mathbb{Z}^p) = 0$ for $i \geqslant 3$, which is immediate from the sequence (*), the acyclicity of $\mathbb{Z}^{p+1}$ and (c). Part (e) also drops out of (*) since $H_1(\mathbb{Z}^{p+1}) = I_{p+1}(\phi)$. (Remember that in taking the exact homology sequence associated to (*) there are dimension shifts.) This completes the proof of Proposition 5.5.

Our goal is to compute the homology of $\mathbb{X}^p(1)$. We know it for $i \geqslant 3$, so we now compute $H_3(\mathbb{X}^p(1))$.

LEMMA 5.6. *In the sequence* 5.5(e), $H_3(\mathbb{Z}^p)$ contains $I_{p+1}^2(\phi)$.

Proof. If we can show that the annihilator of $H_2(\mathbb{U}^p)$ contains $I_{p+1}(\phi)$, then $I_{p+1}^2(\phi)$ is in the kernel of the map of $I_{p+1}(\phi)$ to $H_2(\mathbb{U}^p)$ in 5.5(e). But $H_3(\mathbb{Z}^p)$ is the kernel of this map, so that would prove the lemma.

A simple calculation shows that $H_1(\mathbb{U}^p) = \Lambda^{m-p}N$, where N is the kernel of the generic map $\phi: G \to F^*$. Therefore we have the exact sequence

$$0 \to H_2(\mathbb{U}^p) \to \Lambda^{m-p}N \otimes \Lambda^{p+1}G \otimes G \to \Lambda^{m-p}N \otimes \Lambda^p G.$$

But by [7] we know that $\mathrm{Ann}(\Lambda^{m-p}N)$ contains $I_{p+1}(\phi)$ when ϕ is generic. Therefore the annihilator of $H_2(U^p)$ contains $I_{p+1}(\phi)$ and we are done.

DEFINITION 5.7. The Durfee square 2 complex $\mathbb{X}^p(2)$ is defined to be $\mathbb{Y}(2, F, G)$ of Definition 5.3. That is, $\mathbb{X}^p(2) = \{X_{k+1}^p(2, F, G), \partial(2)\}$, where

$$X_{k+1}^p(2, F, G) = Y_{k+1}(2, F, G)$$

for $k \geqslant 0$, and $\partial(2)$ is the boundary map of $\mathbb{Y}(2, F, G)$.

PROPOSITION 5.8. *There exists a map of complexes*

$$\psi = \{\psi_k : X^p_{k+1}(s, F, G) \to X^p_{k+3}(1, F, G)\}, \qquad k \geqslant 0,$$

such that the composition

$$I^2_{p+1}(\phi) = H_1(\mathbb{X}^p(2)) \xrightarrow{\psi_1^*} H_3(\mathbb{X}^p(1)) = H_3(\mathbb{Z}^p) \to I_{p+1}(\phi)$$

is the inclusion, where ψ_1^* is the map induced by ψ_1 on homology, and $H_3(\mathbb{Z}^p) \to I_{p+1}(\phi)$ is the map of 5.5(e).

Proof. We know by Theorem 5.4 that $\mathbb{X}^p(2)$ is a free acyclic complex, and $H_1(\mathbb{X}^p(2)) = I^2_{p+1}(\phi)$.

We also know that $H_1(\mathbb{X}^p(1)) = 0$ for $i > 3$. If we let $K = \operatorname{Ker}(X^p_3(1, F, G) \to X^p_2(1, F, G))$, we have the exact sequence

$$\begin{aligned} \cdots &\to X^p_{k+1}(1, F, G) \to \cdots \to X^p_4(1, F, G) \to K \\ &\to H_3(\mathbb{X}^p(1)) \to 0. \end{aligned} \tag{***}$$

By Lemma 5.6, we have $I^2_{p+1}(\phi) \subset H_3(\mathbb{X}^p(1))$ embedded so that the composition $I^2_{p+1}(\phi) \subset H_3(\mathbb{X}^p(1)) \to I_{p+1}(\phi)$ is the inclusion. Thus by the comparison theorem there exists a map (unique up to homotopy) from $\mathbb{X}^p(2)$ to (***) which covers the map of $I^2_{p+1}(\phi)$ into $H_3(\mathbb{X}^p(1))$. Since $K \subset X^p_3(1, F, G)$, we may replace K by $X^p_3(1, F, G)$ to obtain the map ψ of our proposition.

Since the map $\psi_1 : X^p_1(2) \to X^p_3(1)$ factors through k, the composition $X^p_1(2) \to X^p_3(1) \to X^p_2(1)$ is zero. Hence we have the map of complexes:

$$\begin{tikzcd} \cdots \arrow[r] & X^p_2(2) \arrow[r] \arrow[d, "\psi_2"] & X^p_2(2) \arrow[r] \arrow[d, "\psi_1"] & 0 \arrow[r] \arrow[d] & 0 \arrow[d] \\ \cdots \arrow[r] & X^p_4(1) \arrow[r] & X^p_s(1) \arrow[r] & X^p_2(1) \arrow[r] & X^p_1(1) \end{tikzcd}$$

and we may take its mapping cone.

DEFINITION 5.9. The mapping cone of the above map will be denoted by $\mathbf{X}^p$.

Remark 5.10. Since the maps ψ_k are not canonically defined, the notation $\mathbb{X}^p$ is, perhaps misleading. However, we saw that these maps ψ_k are unique up to homotopy and, as we will now see, the choice of the ψ_k does not affect our main result.

THEOREM 5.11. (a) *The complex* $\mathbb{X}^p$ *is a minimal free resolution of* $I_p(\phi)$.

(b) $H_3(\mathbb{X}^p(1)) = I^2_{p+1}(\phi)$; $H_2(\mathbb{X}^p(1)) = 0$.

(c) *The Betti numbers of* $I_p(\phi)$ *are independent of characteristic.*

Proof. Once we have established (a), parts (b) and (c) will follow. For clearly the acyclicity of $\mathbb{X}^p$ implies that $H_2(\mathbb{X}^p(1)) = 0$ since $(\mathbb{X}^p)_2 = (\mathbb{X}^p(1))_2$. Also, since the image of ψ_1 in $H_3(\mathbb{X}^p(1))$ is $I^2_{p+1}(\phi)$, the exactness of $\mathbb{X}^p$ in dimension 3 implies that $I^2_{p+1}(\phi)$ must be all of $H_3(\mathbb{X}^p(1))$. Part (c) follows from (a) because of the universality of the complex $\mathbb{X}^p$. That is, if we let $\mathbb{X}^p_z$ be the complex over the ring of integers, then we may take $\mathbb{X}^p$ over any ring R to be $\mathbb{X}^p_z \otimes R$. Thus, the minimality of $\mathbb{X}^p_z$ ensures the invariance of the Betti numbers.

Now for the proof of (a). If we augment the complex $\mathbb{X}^p$ by mapping $X^p_1(1) = \Lambda^p F \otimes \Lambda^p G$ to R, what we want to show is that this augmented complex is a resolution of $R/I_p(\phi)$. The augmented complex has length $2(m-p+2)$ (by Corollary 4.13), so by the acyclicity lemma it suffices to localize at primes of height less than $2(m-p+2)$. (Because of universality, we may assume that R is Cohen–Macaulay so that we need not distinguish between height and grade of ideals.) However, $I_p(\phi)$ is of height $2(m-p+1)$, so that under localization by such a prime, $I_p(\phi)$ blows up. It therefore suffices to prove acyclicity after inverting a $p \times p$ minor of ϕ and, in this case, we may assume that $\phi = id + \phi^1 : H \oplus G^1 \to H \oplus F^{1*}$ where rank $H = p$, rank $F^1 = m - p$, rank $G^1 = 1$, and $\phi^1 : G^1 \to F^{1*}$ is generic. From 5.5(e) we see that it is enough to show, in this case, that $H_2(\mathbb{U}^p) = I_{p+1}(\phi)/I^2_{p+1}(\phi)$, since the map $I_{p+1}(\phi) \to H_2(\mathbb{U}^p)$ will be easily seen to be the canonical surjection. By the usual argument reducing $\mathbb{U}^p$ modulo homotopy equivalence, we may assume that

$$\mathbb{U}^p = (\ \to \Lambda^2 F^1 \to F^1 \xrightarrow{\phi^{1*}} R) \otimes (F^1 \xrightarrow{\phi^{1*}} R),$$

where $I_{p+1}(\phi) = I_1(\phi^1) = (X_{p+1}, \ldots, X_m)$, and $X_{p+1}, \ldots, X_m$ is a regular sequence. The left-hand complex is a Koszul complex on $X_{p+1}, \ldots, X_m$ and a simple argument shows that

$$0 \to H_2(\mathbb{U}^p) \to R/I_1(\phi^1) \otimes F^1 \xrightarrow{1 \otimes \phi^1} R/I_1(\phi^1) \otimes R$$

is exact (we have $H_2(\mathbb{U}^p)$ instead of $H_1(\mathbb{U}^p)$ because of our indexing convention on the complex $\mathbb{U}^p$). But $1 \otimes \phi^1$ is the zero map, and it is well known that $F^1/I_1(\phi^1) \approx I^2_1(\phi^1)$ when ϕ^1 is a regular sequence. This proves the acyclicity of our complex $\mathbb{X}^p$.

All that remains to show now is the minimality of the complex. Since we are not looking over a local, but over a graded ring, by *minimality* we mean that the coefficients of the boundary maps of $\mathbb{X}^p$ are in the ideal of $S(F \otimes G)$ generated by $F \otimes G$. We have already seen that the complexes $\mathbb{X}^p(2)$ and $\mathbb{X}^p(1)$ are linear, so the only maps that need be examined are the maps ψ_k. To do this, recall that all our modules, complexes, etc., are graded, and so, therefore, is the homology of all of these complexes. Looking at the exact sequence of Proposition 5.5(e) from this point of view, we see that $H_3(\mathbb{Z}^p) = 0$ in degrees less than p (since $I_{p+1}(\phi)$ has its first non-zero component in degree $p+1$, and the map $H_3(\mathbb{Z}^p) \to I_{p+1}(\phi)$ is of degree 1). This permits us to define ψ_1 as a map of degree p, and the degrees of the other are therefore also seen to be p.

Remark 5.12. The complex $\mathbb{X}^p(2)$ can be defined in the following way by generators and relations:

$$X_1^p(2) = L_{(p+1)^2}F \otimes L_{(p+1)^2}G \qquad (\dim G = p+1),$$
$$X_2^p(2) = \mathrm{Im}(\Lambda^{p+2}F \otimes \Lambda^{p+1}F \otimes \Lambda^{p+1} \otimes \Lambda^{p+1}G \otimes G \to X_1^p(2) \otimes F \otimes G).$$

In general, we have the exact sequence

$$0 \to X_{k+1}^p(2) \to X_k^p(2) \otimes F \otimes G \to X_{k-1}^p(2) \otimes S_2(F \otimes G),$$

i.e., $X_{k+1}^p(2)$ is the intersection

$$X_1^p(2) \otimes \Lambda^k(F \otimes G) \cap X_2^p(2) \otimes \Lambda^{k-1}(F \otimes G),$$

where each of the above modules is contained in $X_1^p(2) \otimes (F \otimes G) \otimes \Lambda^{k-1}(F \otimes G)$, the first by diagonalizing $\Lambda^k(F \otimes G)$ and the second obviously so.

$X_{k+1}^p(2)$ is generated inside $X_1^p(2) \otimes \Lambda^k(F \otimes G)$ by the images of morphisms ϕ_{a_1,a_2} $(a_1 + a_2 = k,\ a_1 \geqslant a_2)$, where

$$\phi_{a_1,a_2} : \Lambda^{p+1+a_1}F \otimes \Lambda^{p+1+a_2}F \otimes \Lambda^{p+1}G \otimes \Lambda^{p+1}G \otimes D_{a_1}G$$
$$\otimes D_{a_2}G \to X_1^p(2) \otimes \Lambda^k(F \otimes G)$$

is obtained by diagonalizing $\Lambda^{p+1+a_1}F \otimes \Lambda^{p+1+a_1}F$ to $\Lambda^{p+1}F \otimes \Lambda^{p+1}F \otimes \Lambda^{a_1}F \otimes \Lambda^{a_2}F$, using the embeddings of $\Lambda^{a_i}F \otimes D_{a_i}G$ into $\Lambda^{a_i}(F \otimes G)$ for $i = 1, 2$, and multiplying into $\Lambda^k(F \otimes G)$.

The maps ϕ_{a_1,a_2} satisfy the following relations:

$$\begin{tikzcd}
\Lambda^{p+1+a_1+s}F \otimes \Lambda^{p+1+a_2-s}F \otimes \Lambda^{p+1}G \otimes \Lambda^{p+1}G \otimes D_{a_1}G \otimes D_{a_2}G \arrow[r, "1\otimes m\Delta"'] \arrow[d, "m\Delta\otimes 1"] & \Lambda^{p+1+a_1+s}F \otimes \Lambda^{p+1+a_2-s}F \otimes \Lambda^{p+1}G \otimes \Lambda^{p+1}G \otimes D_{a_1+s}G \otimes D_{a_2-s}G \arrow[d, "\Phi_{a_1+s,a_2-s}"] \\
\Lambda^{p+1+a_1}F \otimes \Lambda^{p+1+a_2}F \otimes \Lambda^{p+1}G \otimes \Lambda^{p+1}G \otimes D_{a_1}G \otimes D_{a_1}G \arrow[r, "\Phi_{a_1,a_2}"'] & X_1^p(2) \otimes \Lambda^k(F \otimes G)
\end{tikzcd}$$

and

$$\begin{tikzcd}
\Lambda^{p+1+a_1}F \otimes \Lambda^{p+1+a_2}F \otimes \Lambda^{p+1}G \otimes \Lambda^{p+1}G \otimes D_{a_1+s}G \otimes D_{a_2-s}G \arrow[r] \arrow[d] & \Lambda^{p+1+a_1}F \otimes \Lambda^{p+1+a_2}F \otimes \Lambda^{p+1}G \otimes \Lambda^{p+1}G \otimes D_{a_1}G \otimes D_{a_2}G \arrow[d, "\Phi_{a_1,a_2}"] \\
\Lambda^{p+1+a_1+s}F \otimes \Lambda^{p+1+a_2-s}F \otimes \Lambda^{p+1}G \otimes \Lambda^{p+1}G \otimes D_{a_1+s}G \otimes D_{a_2-s}G \arrow[r, "\Phi_{a_1+s,a_2-s}"'] & X_1^p(2) \otimes \Lambda^k(F \otimes G)
\end{tikzcd}$$

where m indicates multiplication in the appropriate algebra, and Δ a suitable diagonalization. These relations show that

$$\operatorname{Im} \phi_{a_1,a_2} \text{ modulo } \sum_{\substack{b_1>a_1\\ b_1+b_2=k}} \operatorname{Im} \phi_{b_1,b_2}$$

is a quotient of $L_{p+1+a_1,p+1+a_2}F \otimes K_{a_1,a_2}G$. The proof of Theorem 5.4 actually shows equality. Again, a more detailed discussion of generators and relations for $L_\lambda F$ or $K_\lambda F$ will be found in [2].

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